

FAILURE UNDER ALTERNATING LOADS

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J. J. NOLAN

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FAILURE UNDER ALTERNATING LOADS

by

John Jerome Nolan  
Lieutenant Commander, United States Navy

Submitted in partial fulfillment  
of the requirements  
for the degree of  
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THE HISTORY OF THE

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John Jaynes Holms  
Assistant Surgeon, United States Army

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ANATOMICAL SCIENCE

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CONFIDENTIAL

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Monterey, California

June 1955

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### TABLE OF SYMBOLS

$b$	Width of section at neutral axis
$c$	Distance from neutral axis to outermost fiber
$d$	Diameter
$E$	Modulus of elasticity
$e_1 e_2 e_3$	Principal strains
$I$	Moment of inertia
$I_p$	Polar moment of inertia of cross section
$K$	Theoretical stress concentration factor
$k$	Fatigue stress concentration factor
$M$	Bending moment
$M_t$	Torque
$p$	Uniform Tension
$Q$	First Moment of the area about neutral axis
$q$	Sensitivity index
$r$	Radii

# TABLE IV

Width of section at neutral axis	b
Distance from neutral axis to outermost fiber	e
Distance	d
Moment of elasticity	M
Principal stresses	$\sigma_1, \sigma_2$
Moment of inertia	I
Polar moment of inertia of cross section	J
Theoretical stress concentration factor	K
Relaxed stress concentration factor	K
Bending moment	M
Torque	T
Twist	$\theta$
Twist constant of the wire about neutral axis	C
Shear stress	$\tau$
Shear	V



$U$	Strain energy per unit volume
$v$	Shear Force
$V_0$	Distortion energy per unit volume
$\alpha$	Equivalent stress
$\theta$	Phase angle
$\mu$	Poisson's ratio
$\bar{\sigma}$	Mean normal stress
$\sigma_v$	Variable normal stress
$\sigma_1, \sigma_2, \sigma_3$	Principal stresses
$\sigma_m$	Mean stress
$\sigma_{max}$	Maximum stress
$\sigma_{min}$	Minimum stress
$\sigma_c$	Endurance or fatigue limit
$\sigma_{yp}$	Yield point stress
$\sigma_u$	Ultimate tensile stress
$\sigma_1', \sigma_2', \sigma_3'$	Principal stresses, maximum value
$\sigma_1'', \sigma_2'', \sigma_3''$	Principal stresses, mean value



- $T$  Mean shear stress
- $T_v$  Variable shear stress
- $T_{oct}$  Octahedral shearing stress
- $T_s$  Shearing stress

General Fund	7
Special Fund	7
Capital Fund	100
Reserve Fund	12

2

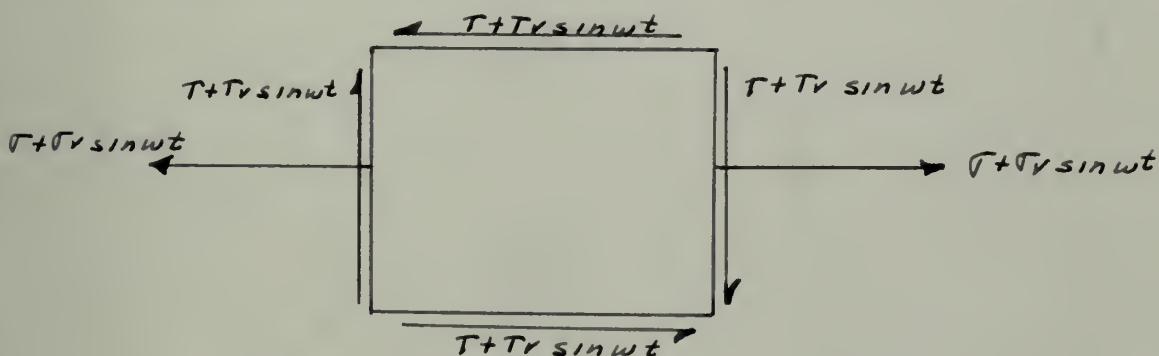
CHAPTER I  
INTRODUCTION

The object of this thesis is to acquire a working knowledge of the principles used in the design of rotating shafts subjected to combined fatigue stress.

The treatment is divided into two parts: first, an investigation of the problems solely from the standpoint of stress; second, a discussion of the effects of stress concentration, hardness of material, and surface treatment in the determination of working stress.

The treatment will be restricted in that:

1. Only axial and shear stresses will be considered, as shown for an element in the diagram below.
2. Materials will be regarded as ductile, isotropic, and homogeneous.



In arriving at the general relations for the allowable working stress in shafting, two sub-cases will be discussed:



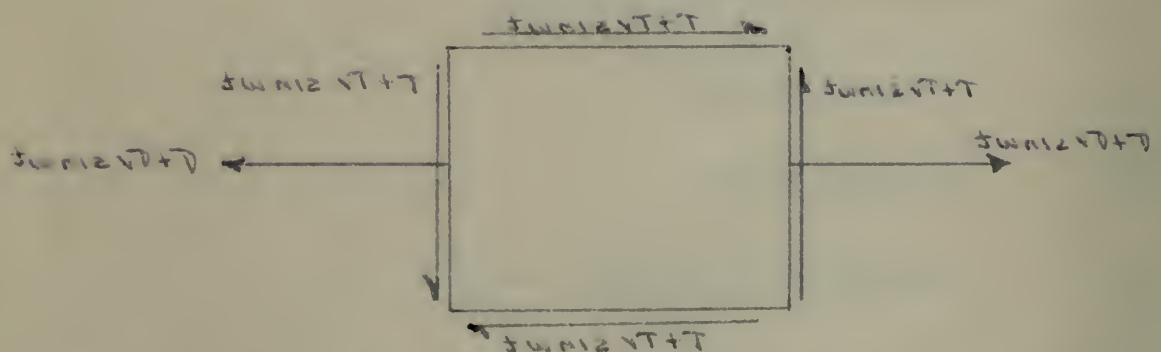
# CHAPTER I INTRODUCTION

The object of this thesis is to acquire a working knowledge of the principles used in the design of rotating shafts subjected to combined fatigue stress.

The treatment is divided into two parts: first, an investigation of the problems solely from the standpoint of stress; second, a discussion of the effects of stress concentration, hardness of material, and various treatments in the determination of working stress.

The treatment will be restricted in that:

1. Only axial and shear stresses will be considered, as shown for an element in the diagram below.
2. Materials will be regarded as isotropic, isotropic, and homogeneous.



In arriving at the general relations for the allowable working stress in rotating, two sub-cases will be discussed:

(a) Axial stress varying between maximum and minimum values while the shear stress remains constant.

(b) Axial stress and shear stress varying between maximum and minimum values of different magnitudes and in phase.

Note that the axial and shear stresses can be computed using the standard formulas:  $\frac{Mc}{I}$ ,  $\frac{P}{A}$ ,  $\frac{M_{tr}}{I_p}$ ,  $\frac{VQ}{Ib}$ .





## CHAPTER II

### THEORIES OF FAILURE

For the case of static stresses, the allowable stress can be determined in terms of the principal stresses using one of the theories of failure. In discussing the various theories, only those which are in near agreement with actual tests will be presented as they not only predict with more accuracy failure from static combined stresses, but also failure due to fluctuating stresses. Failure is defined as the beginning of inelastic action (yielding).

The treatment of the theories of failure will consider the most general case, that is, the stress condition of an element of a body is defined by the magnitudes of the principal stresses  $\sigma_1, \sigma_2$  and  $\sigma_3$ . For convenience, we presume  $\sigma_1 \geq \sigma_2 \geq \sigma_3$ .

The Maximum Shear Theory, as first suggested by Coulomb, assumes that failure of materials subjected to combined stresses is due to shear rather than direct stress. To lend support to this assumption, the physical appearance of material after being subjected to load reveals the presence of so-called slip layers of fine markings on the surface of the deformed bodies which approximately coincide with the planes of maximum shearing stress. As Nádai (1) points out, the fine line markings were interpreted as the intersections of thin layers of material with the surface of the deformed pieces, in which the grain structure of the substance was distorted through the yielding. These planes in certain materials are inclined at an angle of 45 degrees with respect to the directions

## CHAPTER II THEORY OF FAILURE

For the case of simple stresses, the elastic limit has been determined in terms of the principal stresses and the theories of failure. In discussing the various theories, only those which are in best agreement with actual tests will be presented as they are only tested with simple failure tests static combined stresses, but also failure due to fluctuating stresses. Failure is defined as the beginning of plastic action (yielding).

The treatment of the theories of failure will consider the most general case, that is, the general condition of an element of a body is defined by the magnitudes of the principal stresses  $\sigma_1, \sigma_2$  and  $\sigma_3$ . For convenience, we assume  $\sigma_1 \geq \sigma_2 \geq \sigma_3$ . The Maximum Shear Theory, as first suggested by Coulomb, assumes that failure of materials subjected to combined stresses is due to shear rather than direct stress. To lend support to this assumption, the physical appearance of materials after being subjected to load reveals the presence of so-called shear layers of lines extending on the surface of the deformed bodies which approximately coincide with the planes of maximum shearing stress. As Mohr (1) points out, the line elements were interpreted as the intersections of thin layers of material with the surface of the deformed plates, in which the grain structure of the substance was distorted through the yielding. These planes in deformed materials are inclined at an angle of 45 degrees with respect to the directions



of the largest and smallest principal stress. Based on this assumption, Guest later formulated the maximum shear theory. By this theory, failure occurs when the maximum shear stress in an element subjected to combined stresses reaches the value of the maximum shear stress at failure in simple tension.

It may be shown that the extreme shearing stress occurs on planes bisecting the dihedral angles between the principal planes. The magnitudes are

$$\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2 - \sigma_3}{2}, \text{ and } \frac{\sigma_1 - \sigma_3}{2}$$

Because of the convention adopted above, the greatest shearing stress is  $\tau = \frac{\sigma_1 - \sigma_3}{2}$

Since from the limiting case of simple tension or compression, the maximum shear stress becomes,  $\tau = \sigma/2$ . The maximum shear theory predicts failure will occur when  $(\sigma_1 - \sigma_3)/2$  becomes equal to the shear at failure in simple tension, or

$$\sigma_1 - \sigma_3 = \sigma$$

The Maximum Strain Energy Theory, as suggested by Beltrami, later formulated by Huber, and still later again by Haigh, predicts that failure is based on the concept of energy of deformation. It assumes that failure results when the total strain energy of deformation per unit volume, in the case of combined stresses, is equal to the strain energy per unit volume in simple tension. For gradually applied loads, the strain energy per unit volume is

$$U = \frac{\sigma_1 e_1}{2} + \frac{\sigma_2 e_2}{2} + \frac{\sigma_3 e_3}{2}$$

where  $e_1$ ,  $e_2$ , and  $e_3$  are the principal strains. On substituting the values for strains in terms of principal stresses, the strain

of the largest and smallest principal stresses, based on the  
 assumption, that these two principal stresses are equal. By  
 this theory, failure occurs when the maximum shear stress is at  
 a point subjected to combined stresses. The value of the  
 maximum shear stress at failure is denoted by  $\tau_{max}$ .  
 It may be shown that the maximum shear stress occurs on  
 planes inclined at 45 degrees to the principal stresses.

The maximum stress

$$\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_1 + \sigma_2}{2}, \text{ and } \frac{\sigma_1 - \sigma_2}{2}$$

Because of the convention adopted above, the maximum shearing

$$\tau = \frac{\sigma_1 - \sigma_2}{2}$$

Since from the preceding class of single tension or compression,  
 the maximum shear stress occurs,  $\tau = \frac{\sigma}{2}$ .  
 The maximum shear stress occurs at failure when  $\tau = \tau_{max}$ .  
 becomes equal to the shear at failure in simple tension, or

$$\tau = \tau_{max}$$

The maximum stress,  $\sigma_1$ , is denoted by  $\sigma_u$ .  
 later is denoted by  $\sigma_u$ , and still later again by  $\sigma_u$ , etc.  
 since this failure is based on the concept of energy at deformation.  
 It assumes that failure occurs when the total strain energy of  
 deformation per unit volume, in the case of combined stresses, is  
 equal to the strain energy per unit volume in simple tension. For  
 gradually applied loads, the strain energy per unit volume is

$$U = \frac{\sigma_1 \epsilon_1}{2} + \frac{\sigma_2 \epsilon_2}{2} + \frac{\sigma_3 \epsilon_3}{2}$$

where  $\sigma_1, \sigma_2$  and  $\sigma_3$  are the principal stresses. On substitution  
 the value for strain in terms of principal stresses, the strain

energy per unit volume becomes

$$U = \frac{1}{2E} [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu (\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1)]$$

where E is the modulus of elasticity and  $\mu$  is Poisson's ratio.

Since at the beginning of elastic failure in simple tension, the unit strain is  $\sigma/E$ , the strain energy for simple tension becomes  $\sigma^2/2E$ . The theory then predicts that failure will occur when

$$U = \frac{1}{2E} [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu (\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1)] = \frac{\sigma^2}{2E}$$

The Maximum Distortion Energy Theory was developed by Von

Mises and Hencky. It assumes that failure begins in the case of combined stress when the energy of distortion or shear approaches the same energy at failure as in the case of simple tension. In the development of this theory, it is considered that the total strain energy (U) is divided into two parts: the energy to produce a change in volume and the energy used to distort the element. Only the second part is used in the development of this theory. The theory was brought about by the fact that isotropic materials can endure large hydrostatic pressures without yielding. To develop the theory, we first divide the principal stresses in two parts

$$\sigma_1 = \sigma_1' + p, \quad \sigma_2 = \sigma_2' + p, \quad \sigma_3 = \sigma_3' + p$$

where

$$p = \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3)$$

Since from this theory

$$\sigma_1' + \sigma_2' + \sigma_3' = 0$$

the stress condition  $\sigma_1', \sigma_2', \sigma_3'$ , produces only distortion and the change in volume depends entirely on the magnitude of the uniform tension (p), the part of the total energy due to a change in volume is

$$\frac{ep}{2} = \frac{3(1-2\mu)p^2}{2E} = \frac{1-2\mu}{6E} (\sigma_1 + \sigma_2 + \sigma_3)^2$$



energy per unit volume

$$U = \frac{1}{2} \epsilon_0 \int_V (\nabla \cdot \mathbf{E})^2 dV = \frac{1}{2} \epsilon_0 \int_V (\nabla^2 \phi)^2 dV$$

where  $\epsilon_0$  is the permittivity of free space and  $\phi$  is the electrostatic potential.

Since at the beginning of the process the system is in a state of minimum energy, the initial energy is  $U_0$ . The final energy is  $U_f$  and the energy difference  $U_f - U_0$  is the energy that has been converted into other forms.

$$U = \frac{1}{2} \epsilon_0 \int_V (\nabla \cdot \mathbf{E})^2 dV = \frac{1}{2} \epsilon_0 \int_V (\nabla^2 \phi)^2 dV$$

The energy difference  $U_f - U_0$  is the energy that has been converted into other forms.

Mass and energy. It appears that energy begins in the case of combined stress when the energy of distortion or shear is expended the same energy as before is in the case of static equilibrium. In the development of this theory, it is considered that the total static energy (U) is divided into two parts: the energy of volume stress in volume and the energy due to distort the element.

Only the second part is used in the development of this theory.

The theory was proposed based on the fact that isotropic materials can endure large hydrostatic pressures without yielding. To develop the theory, we first divide the principal stresses in two parts:

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_0 + \sigma_1'$$

where

$$\sigma_1' = \frac{1}{2} (\sigma_1 - \sigma_2)$$

Since the total theory

$$U = \frac{1}{2} \epsilon_0 \int_V (\nabla \cdot \mathbf{E})^2 dV = \frac{1}{2} \epsilon_0 \int_V (\nabla^2 \phi)^2 dV$$

the stress condition  $\sigma_1' = \sigma_2' = \sigma_3' = 0$  is satisfied only distortion and the

change in volume depends entirely on the magnitude of the volume

stress (p), the value of the total energy due to a change in volume

$$\frac{\partial U}{\partial \sigma_0} = \frac{1}{2} \epsilon_0 \int_V (\nabla \cdot \mathbf{E})^2 dV = \frac{1}{2} \epsilon_0 \int_V (\nabla^2 \phi)^2 dV$$

where  $e = e_1 \neq e_2 \neq e_3$ . Subtracting the total energy due to a change in volume from the total strain energy as determined and using the identity

$$\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1 = -\frac{1}{2} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] + \sigma_1^2 + \sigma_2^2 + \sigma_3^2$$

the part of the total strain energy due to distortion can be presented in the form

$$V = U - \frac{1-2\mu}{E} (\sigma_1 + \sigma_2 + \sigma_3)^2 = \frac{1+\mu}{6E} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 \right]$$

The distortion energy at failure in simple tension is obtained by placing  $\sigma_2 = \sigma_3 = 0$  and  $\sigma_1 = \sigma$  or

$$V = \frac{(1+\mu)}{3E} \sigma^2$$

For combined stress, we then have

$$V = \frac{1}{2} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 \right] = \sigma^2$$

It is of interest to note that the expression within the brackets namely,

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2$$

is proportional to the square of the shearing stress on an octahedral plane (planes whose normals have direction cosines  $\pm 1/\sqrt{3}$  with respect to the principal directions.)

Since the expression for the octahedral shearing stress is

$$\tau_{oct} = \frac{1}{3} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}$$

the condition for failure may be expressed as

$$\tau_{oct} = \frac{\sqrt{2}}{3} \sigma$$

where  $x = y = z = 0$ . Substitution of the total energy and the  
 a change in volume from the total kinetic energy as determined

and using the identity

$$(\sqrt{1-x^2} + \sqrt{1-y^2} + \sqrt{1-z^2})^2 = 1 + x^2 + y^2 + z^2 + 2(\sqrt{1-x^2}\sqrt{1-y^2} + \sqrt{1-x^2}\sqrt{1-z^2} + \sqrt{1-y^2}\sqrt{1-z^2})$$

the part of the total kinetic energy due to vibration can be

presented in the form

$$V = \sqrt{1-x^2} + \sqrt{1-y^2} + \sqrt{1-z^2} + \frac{1}{2} \left[ (\sqrt{1-x^2}\sqrt{1-y^2} + \sqrt{1-x^2}\sqrt{1-z^2} + \sqrt{1-y^2}\sqrt{1-z^2}) \right]$$

The vibration energy of lattice in which beam is ob-

ained by dividing  $V = 1$  and  $V = 1$  or

$$V = \frac{1}{2} \left[ \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}} + \frac{1}{\sqrt{1-z^2}} \right]$$

For constant stress, we have

$$V = \frac{1}{2} \left[ (\sqrt{1-x^2})^2 + (\sqrt{1-y^2})^2 + (\sqrt{1-z^2})^2 \right] = V^2$$

It is of interest to note that the expansion of this the

bracket term,

$$(\sqrt{1-x^2})^2 + (\sqrt{1-y^2})^2 + (\sqrt{1-z^2})^2$$

is proportional to the square of the elliptical integral on an elliptical

plane (planes whose curves have constant modulus  $\pm \sqrt{1/2}$ )

with respect to the principal directions.)

Since the expression for the vibrational energy across is

$$1/2 + \frac{1}{2} \sqrt{(1-x^2)^2 + (1-y^2)^2 + (1-z^2)^2}$$

the condition for failure may be expressed as

$$1/2 + \frac{1}{2} \sqrt{(1-x^2)^2 + (1-y^2)^2 + (1-z^2)^2} = 1$$



### CHAPTER III

#### SIMPLE FATIGUE STRESS

With the static criteria of failure established, a discussion of the manner in which an expression is determined for failure of materials subjected to simple axial fatigue stress is next presented, since both concepts are used in the development of theories of failure for materials subjected to combined fatigue stress.

For simple axial fatigue stress, the variation of stress with time is usually sinusoidal. It may be represented as in Figure 1 (see separate sheet), and expressed as:

$$\sigma = \sigma_m + \sigma_v \sin \omega t$$

The representation further shows that the values of the mean stress  $\sigma_m$  and the variable stress  $\sigma_v$  are

$$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2} \quad , \quad \sigma_v = \frac{\sigma_{max} - \sigma_{min}}{2}$$

The stress is thus conveniently thought of as consisting of two parts: a reversed stress superimposed on a steady, or static, stress.

With this in mind, two limiting conditions of stress at failure are possible. For the first condition,  $\sigma_v = 0$ , the stress is entirely static and failure occurs when  $\sigma_v = \sigma_{yp}$  (the yield point in simple tension or compression). For the second condition,  $\sigma_m = 0$ , failure results from complete reversal of stress repeated a large number of times. From this type of failure, the endurance limit or fatigue limit  $\sigma_e$  of a material is obtained.

# CHAPTER III

## STRESS RELATIONSHIPS

With the elastic behavior of materials established, a discussion of the manner in which an extension is determined for failure of materials subjected to simple axial fatigue stress is next presented, since both concepts are used in the development of theories of failure for materials subjected to combined fatigue stresses.

For simple axial fatigue stress, the variation of stress with time is usually sinusoidal. It may be represented as in Figure 1 (see separate sheet), and expressed as:

$$\sigma = \sigma_m + \sigma_a \sin \omega t$$

The representation shows that the value of the mean stress  $\sigma_m$  and the variation  $\sigma_a$  are

$$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2}, \quad \sigma_a = \frac{\sigma_{max} - \sigma_{min}}{2}$$

The stress  $\sigma$  then conveniently consists of an oscillating or alternating stress superimposed on a steady, or static, stress.

With this in mind, the limiting conditions of stress at failure are possible. For the first condition,  $\sigma_a = 0$ , the stress is entirely static and failure occurs when  $\sigma = \sigma_y$  (the yield point in static tension or compression). For the second condition,  $\sigma_m = 0$ , failure occurs from complete reversal of stress repeated a large number of times. For this type of failure, the endurance limit or fatigue limit  $\sigma_e$  of a material is obtained.







The endurance limit is considered as the maximum stress which a member can sustain for an indefinitely large number of cycles (usually taken as at least 10 million cycles for ferrous materials and about 50 million cycles for non-ferrous materials). Fatigue limits have been determined for various types of simple stresses such as alternating tension and compression, bending, and torsion for most of the materials in use today. It is of interest to note that the value of  $\sigma_e$ , as determined from tests of materials, Timoshewko (2), Moore and Koomers (3), is appreciably less than the value of  $\sigma_{yp}$  for the same material, thus the resisting strength of materials is reduced under the conditions of variable stress.

Since most problems present conditions intermediate between these extremes, it is necessary to consider all possible combinations of maximum and minimum stress or, more properly, it is necessary to consider the effect of mean stress on fatigue strength. A great deal of information can be obtained for variable axial stress and a typical diagram (Figure 2, see separate sheet) shows endurance limit ratios for a number of combinations of axial fluctuating stress, ( $\sigma_u$  is the ultimate tensile stress.)

Various attempts have been made to interpret such tests. The methods used reduce to empirical equations giving relations between the variable and mean stresses or the maximum stresses at failure. Of the various proposals, three are most used in design and are presented as follows:

GERBER'S LAW - Gerber's law is an empirical relationship which assumes that the relation for defining the variation of the variable stress with mean stress is of the parabolic form,

$$\frac{\sigma_v}{\sigma_e} = 1 - (\sigma_m/\sigma_u)^2$$

The endurance limit is considered as the maximum stress which a member can sustain for an indefinitely large number of cycles (usually taken as at least 10 million cycles for ferrous materials and about 50 million cycles for non-ferrous materials). Fatigue limits have been determined for various types of simple stresses such as alternating tension and compression, bending, and torsion for most of the materials in use today. It is of interest to note that the value of  $\sigma_e$ , as determined from tests of specimens, Timoshenko (2), Moore and Bozars (3), is appreciably less than the value of  $\sigma_{yp}$  for the same material, since the testing strength of materials is reduced under the conditions of certain stresses.

Since most problems present conditions intermediate between these extremes, it is necessary to consider all possible combinations of maximum and minimum stress or, more properly, it is necessary to consider the effect of mean stress on fatigue strength. A great deal of information can be obtained for various axial stress and a typical diagram (Figure 2, see separate sheet) shows endurance limit ratios for a number of combinations of axial fluctuating stress, ( $\sigma_e$  is the ultimate tensile stress).

Various attempts have been made to interpret such tests. The methods used reduce to empirical equations giving relations between the variable and mean stresses at the maximum stresses at failure. Of the various proposals, three are most used in design and are presented as follows:

#### GERBNER'S LAW - Gerbner's law is an empirical relationship

which assumes that the relation for obtaining the variation of the variable stress with mean stress is of the S-N type form,

$$\frac{\sigma}{\sigma_e} = 1 - \left( \frac{\sigma_m}{\sigma_e} \right)^2$$

or, in terms of the variable and mean stresses,

$$\sigma_v = \sigma_e - \sigma_e (\sigma_m / \sigma_u)^2$$

GOODMAN LAW - The Goodman Law is an empirical law which assumes that the relation defining failure for different combinations of variable stress and mean stresses is a straight line between the end points  $\sigma_v / \sigma_e$  and  $\sigma_m / \sigma_u$ . The equation for this straight line is

$$\sigma_v / \sigma_e = 1 - \sigma_m / \sigma_u$$

or, in terms of the variable and mean components,

$$\sigma_v = \sigma_e - (\sigma_m / \sigma_u) \sigma_e$$

STRAIGHT LINE LAW - The Straight Line Law proposed by Soderberg (4) assumes that this relation is defined by a straight line between the end points  $\sigma_v / \sigma_e$  and  $\sigma_{yp} / \sigma_m$

The empirical relation is

$$\sigma_v / \sigma_e = 1 - \sigma_m / \sigma_{yp}$$

or, in a form which will be later used

$$\sigma_{max} = (1 - \sigma_e / \sigma_{yp}) \sigma_m + \sigma_e$$

Of the three empirical relations, the Straight Line Law predicts with more accuracy failure under simple fatigue stress and will be used in the later developments of the combined fatigue stress theories. The major objection to the Gerber and Goodman Laws is that some of the test data falls below the empirical curves (on the unsafe side, indicated in Figure 2).



of, in terms of the variables and mean response,

$$E_y = E_x - \frac{1}{2} \frac{E_{xy}}{E_{xx}}$$

where  $E_{xy}$  is the covariance between  $x$  and  $y$ , and  $E_{xx}$  is the variance of  $x$ .

which assumes that the relation between  $x$  and  $y$  is linear.

consideration of variance ratios and mean response is a standard

line between the two points  $E_x$  and  $E_y$ . The equation for

this straight line is

$$E_y = E_x - \frac{1}{2} \frac{E_{xy}}{E_{xx}}$$

or, in terms of the variables and mean components,

$$E_y = E_x - \frac{1}{2} \frac{E_{xy}}{E_{xx}}$$

STRAIGHT LINE - The straight line has equation of

Regression (a) assumes that this relation is defined by a straight

line between the two points  $E_x$  and  $E_y$ .

The empirical relation is

$$E_y = E_x - \frac{1}{2} \frac{E_{xy}}{E_{xx}}$$

or, in a form which will be later used

$$E_{yx} = (1 - \frac{1}{2} \frac{E_{xy}}{E_{xx}}) E_x + \frac{1}{2} E_y$$

Of the above empirical relations, the straight line has

properties with most accuracy relative to the other two and

will be used in the later development of the combined relation.

straight line. The ratio of  $E_{xy}$  to  $E_{xx}$  and  $E_{yy}$  is the

in that case of the least squares method the empirical curves (or

the unsmoothed data, indicated in Figure 2).

Figure 2 is a plot of

Figure 2 - A plot of

the unsmoothed data and the straight line fit to the data.

Figure 2 is a plot of the unsmoothed data and the straight line fit to the data.



## CHAPTER IV

### COMBINED FATIGUE STRESS

Under the conditions of variable stress, it was shown in the previous chapter that the resisting strength of a material is reduced. In the same way, the resisting strength in the case of combined stress will be modified. Since we are dealing with combined fatigue stress, the determination of working stresses for fluctuation loads will require the development of theories for defining failure as was done for the case of static combined stresses. The development of these theories will be presented in order to develop an accepted expression for the determination of working stress for the case under consideration.

The treatment of the theories of failure will consider the most general case as well as the case under consideration with one restriction, that is, only the case under consideration will be considered for the Shear Theory. For the general case, stress conditions on an elemental body are defined by the principal stresses  $\sigma_1', \sigma_2', \sigma_3'$  and  $\sigma_1'', \sigma_2'', \sigma_3''$  where the prime and double prime notation refer to the maximum and mean values of principal stress, respectively. The notation is similar to that used by Marin (5).

SHEAR THEORY - To develop a shear theory for failure, we first express the equation for defining failure

$$\sigma_{max} = (1 - \sigma_c / \sigma_{yp}) \sigma_m + \sigma_c$$

in terms of fluctuating shearing stresses instead of fluctuation axial stresses.

75 1571415

1001

and "The Prince and the Pauper"

RECEIVED - 10/10/1968

First episode of delirium

$$x + m(1 - x) = x + m - mx$$

... ..

The conversion can be accomplished by replacing  $\tau_{max}$  and  $\tau_m$  by  $\tau_{max} = \tau_{max}/2$  and  $\tau_m = \tau_m/2$  giving the equation

$$\tau_{max} = (1 - \tau_c/\tau_{yp}) \tau_m + \tau_c/2 \quad (a)$$

Since for this theory only normal and shearing stresses will be considered, it is necessary in defining failure to consider the shear stress  $\tau_s$  on any plane as shown for the element in question, Figure 3.

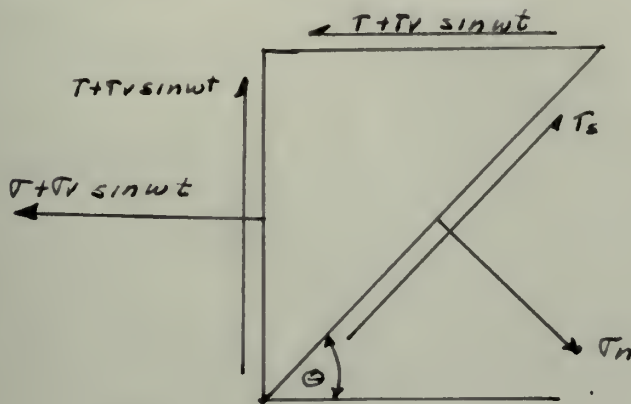


FIG. 3

For this particular plane, the maximum and mean components of shear stress are

$$\tau_{max} = \frac{1}{2} (\sigma + \tau_v) \sin 2\theta + (T + \tau_v) \cos 2\theta$$

$$\tau_m = \frac{1}{2} \sigma \sin 2\theta + T \cos 2\theta$$

Failure for the plane, by the shear theory will occur when the above value of  $\tau_{max}$  and  $\tau_m$  satisfy Eq. a.

Therefore,

$$[\sigma + \tau_v - \sigma(1 - \tau_c/\tau_{yp})] \sin 2\theta = 2[(1 - \tau_c/\tau_{yp})T - (T + \tau_v)] \cos 2\theta + \tau_c$$



The components are the components of the stress  $T_{xx}$  and  $T_{yy}$  of  $T_{xx} = T_{yy} = \frac{1}{2} T$  and  $T_{xy} = T_{yx} = \frac{1}{2} T$

Using the equation

$$T_{xx} = (1 - \cos 2\theta) T_{xx} + \sin 2\theta T_{xy}$$

(2)

Since the stress is only normal, the shear stress will be zero. It is necessary to obtain the value of  $\theta$  for the element in question, Figure 3.

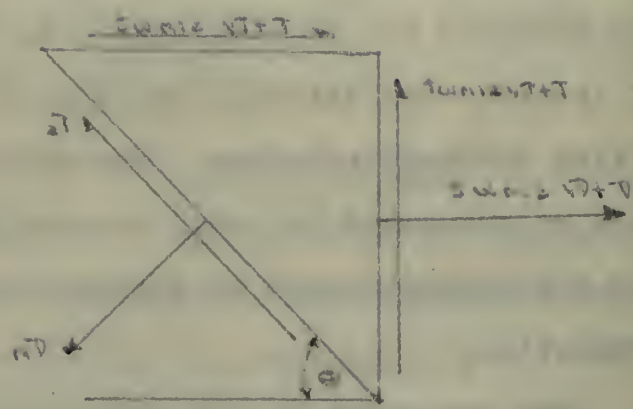


Fig. 3

For this particular case, the normal and shear components of stress are

$$T_{xx} = \frac{1}{2} (T + T) \cos 2\theta + (T + T) \sin 2\theta$$

$$T_{xy} = \frac{1}{2} T \sin 2\theta + T \cos 2\theta$$

Therefore, for the case, the shear stress will occur when the above value of  $T_{xx}$  and  $T_{xy}$  satisfy the

Therefore,

$$[T + T - (1 - \cos 2\theta) T - (1 + \sin 2\theta) T] \sin 2\theta = 2[(1 - \cos 2\theta) T - (1 + \sin 2\theta) T] \cos 2\theta$$



To determine the critical plane on which failure occurs, the above equation can be defined in terms of an equivalent stress  $\alpha$  on any plane such that

$$\alpha = \left[ (\sigma + \tau_v) - \sigma (1 - \sigma_e / \sigma_{yp}) \right] \sin 2\theta - 2 \left[ (1 - \sigma_e / \sigma_{yp}) \tau - (\tau + \tau_v) \right] \cos 2\theta - \sigma_e$$

To find the most critical plane  $\alpha$  should be a maximum. The maximum value of  $\alpha$  occurs when

$$\tan 2\theta = \frac{(1 - \sigma_e / \sigma_{yp}) \tau - (\sigma + \tau_v)}{2 \left[ (\sigma + \tau_v) - (1 - \sigma_e / \sigma_{yp}) \tau \right]}$$

On substituting the value of  $\theta$  into Eq. (b) and combining terms, failure by the stress theory is expressed as

$$\sigma_e = \sqrt{\left[ (\sigma_e / \sigma_{yp}) \tau + \tau_v \right]^2 + 4 \left[ (\sigma_e / \sigma_{yp}) \tau + \tau_v \right]^2}$$

DISTORTION ENERGY THEORY - The assumption to be made in the case of this theory is that failure occurs in the case of combined stresses when the distortion energy corresponding to the maximum value of stress components equals the distortion energy at failure for maximum axial stress. It is also necessary to require that this applies for equal values of distortion energy corresponding to the mean axial and mean combined stresses. To arrive at the above condition, it is necessary to define the fluctuating axial stress in terms of distortion energy, and use the relation as determined from the derivation of combined static stress. The distortion energy at any instant of time for simple axial stress as determined previously is

$$V = \left( \frac{1 + \mu}{3 E} \right) \sigma^2$$

To determine the critical value of  $\lambda$  we set  $\partial W / \partial \lambda = 0$ . The energy equation can be written in terms of  $\lambda$  as follows:

$$W = \frac{1}{2} \left[ (1 + \lambda) - \frac{1}{2} (1 - \lambda) \right] \left[ \frac{1}{2} (1 + \lambda) - \frac{1}{2} (1 - \lambda) \right] \cos \theta$$

To find the most critical value of  $\lambda$  we set  $\partial W / \partial \lambda = 0$ . The maximum value of  $\lambda$  is given by

$$\lambda = \frac{(1 + \lambda) - \frac{1}{2} (1 - \lambda)}{\frac{1}{2} (1 + \lambda) - \frac{1}{2} (1 - \lambda)}$$

On substituting the value of  $\lambda$  into (1) and (2) we obtain the values of  $\lambda$  and  $\theta$  respectively as

$$\lambda = \frac{1}{2} \left[ (1 + \lambda) - \frac{1}{2} (1 - \lambda) \right] + \frac{1}{2} \left[ (1 + \lambda) - \frac{1}{2} (1 - \lambda) \right]$$

Critical value of  $\lambda$  - The assumption is made in the case of this theory is that failure occurs in the case of combined stresses when the distortion energy corresponding to the maximum value of stress components equals the distortion energy at failure for maximum axial stress. It is also necessary to require that the value of axial stress of distortion energy corresponding to the case axial and shear combined stresses. To arrive at the above condition, it is necessary to write the following axial stress in terms of distortion energy, and the relation as follows from the derivation of combined stress theory. The distortion energy at any instant of time for simple axial stress as determined previously is

$$U = \frac{1}{2} \left( \frac{\sigma}{E} \right)^2 V$$

From this equation, the values of distortion energy corresponding to the maximum and mean values of fluctuating stress are

$$V_{max} = \left( \frac{1+\mu}{3E} \right) (\sigma_{max})^2, \quad V_m = \left( \frac{1+\mu}{3E} \right) \sigma_m^2$$

Since our equation for failure is

$$\sigma_{max} = (1 - \sigma_e / \sigma_{yp}) \sigma_m + \sigma_e$$

failure in terms of distortion energies can now be expressed as

$$\sqrt{V_{max}} = (1 - \sigma_e / \sigma_{yp}) \sqrt{V_m} + \left( \sqrt{\frac{1+\mu}{3E}} \right) \sigma_e$$

To get some general relations, the three dimensional case will again be considered since it was determined under theories of failure that distortion energy in terms of static principal stresses is

$$V = \left( \frac{1+\mu}{3E} \right) \left[ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_1 \sigma_3) \right]$$

This generalization would give values for distortion energies corresponding to the maximum and mean components of stresses as

$$V_{max} = \left( \frac{1+\mu}{3E} \right) \left[ (\sigma_1')^2 + (\sigma_2')^2 + (\sigma_3')^2 - (\sigma_1' \sigma_2' + \sigma_2' \sigma_3' + \sigma_1' \sigma_3') \right]$$

$$V_m = \left( \frac{1+\mu}{3E} \right) \left[ (\sigma_1'')^2 + (\sigma_2'')^2 + (\sigma_3'')^2 - (\sigma_1'' \sigma_2'' + \sigma_2'' \sigma_3'' + \sigma_1'' \sigma_3'') \right]$$

Failure by the distortion energy theory is now defined by substituting these expressions into the failure relation such that

$$\sigma_e = \sqrt{(\sigma_1')^2 + (\sigma_2')^2 + (\sigma_3')^2 - (\sigma_1' \sigma_2' + \sigma_2' \sigma_3' + \sigma_1' \sigma_3')} - (1 - \sigma_e / \sigma_{yp}) \sqrt{(\sigma_1'')^2 + (\sigma_2'')^2 + (\sigma_3'')^2 - (\sigma_1'' \sigma_2'' + \sigma_2'' \sigma_3'' + \sigma_1'' \sigma_3'')}$$



From this equation, the values of distortion energy corresponding to the various and mean values of twisting stress are

$$V_{max} = \left( \frac{1+\mu}{3E} \right) (\bar{V}_{max})^2, \quad V_m = \left( \frac{1+\mu}{3E} \right) \bar{V}_m^2$$

Since the slope of the curves is

$$\bar{V}_{max} = (1 - 0.5774) \bar{V}_m = 0.4226 \bar{V}_m$$

Substituting in terms of distortion energy one can be obtained as

$$\bar{V}_{max} = \left( \frac{1+\mu}{3E} \right) (\bar{V}_m)^2 + \left( \frac{1+\mu}{3E} \right) \bar{V}_m^2$$

To get some general relations, the three dimensional case will again be considered since it is understood under the term of failure that distortion energy is taken as a scalar quantity.

$$V = \left( \frac{1+\mu}{3E} \right) \left[ (\sigma_1 + \sigma_2 + \sigma_3)^2 - (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \right]$$

This generalization would give values for distortion energy corresponding to the various and mean values of stresses as

$$V_{max} = \left( \frac{1+\mu}{3E} \right) \left[ (\sigma_1 + \sigma_2 + \sigma_3)^2 - (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \right]$$

$$V_m = \left( \frac{1+\mu}{3E} \right) \left[ (\sigma_1 + \sigma_2 + \sigma_3)^2 - (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \right]$$

Failure by the distortion energy theory is now defined by substituting these expressions into the failure relation such that

$$(\sigma_1 + \sigma_2 + \sigma_3)^2 - (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) = (\sigma_1 + \sigma_2 + \sigma_3)^2 - (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$



For the two dimensional case in which the combined stresses are the principal stresses, the distortion energy theory predicts (on letting  $\sigma_3' = \sigma_3'' = 0$ )

$$\sigma_e = \sqrt{(\sigma_1')^2 + (\sigma_2')^2 - (\sigma_1'\sigma_2')} - (1 - \sigma_e/\sigma_{yp}) \sqrt{(\sigma_1'')^2 + (\sigma_2'')^2 - \sigma_1''\sigma_2''}$$

To express this theory for the two dimensional stress components of Figure 3, it is only necessary to express the principal stresses in the above equations in terms of stress components. The substitution gives

$$\sigma_e = \sqrt{(\sigma + \tau_v)^2 + 3(\tau + \tau_v)^2} + (1 - \sigma_e/\sigma_{yp}) \sqrt{\sigma^2 + 3\tau^2}$$

STRAIN ENERGY THEORY - Using the assumption that failure occurs as a result of total strain energy rather than distortion energy, another theory of failure can be developed. Since the manner of developing this theory is similar to that used for the distortion energy theory, the three dimensional case for strain energy may be expressed as:

$$\sigma_e = \sqrt{(\sigma_1')^2 + (\sigma_2')^2 + (\sigma_3')^2 - 2\mu(\sigma_1'\sigma_2' + \sigma_2'\sigma_3' + \sigma_1'\sigma_3')} - (1 - \sigma_e/\sigma_{yp}) \sqrt{(\sigma_1'')^2 + (\sigma_2'')^2 + (\sigma_3'')^2 - 2\mu(\sigma_1''\sigma_2'' + \sigma_2''\sigma_3'' + \sigma_1''\sigma_3'')}$$

or in the two dimensional case where  $\sigma_3' = \sigma_3'' = 0$ .

$$\sigma_e = \sqrt{(\sigma_1')^2 + (\sigma_2')^2 + 2\mu(\sigma_1'\sigma_2')} - (1 - \sigma_e/\sigma_{yp}) \sqrt{(\sigma_1'')^2 + (\sigma_2'')^2 + 2\mu\sigma_1''\sigma_2''}$$

(a)  $\phi = 0$

דב' (ה'י) + (ה'י) - (ה'י) = (ה'י) - (י-י) (ה'י) + (ה'י) = (ה'י) - (ה'י)

To express this identity for the two dimensional case con-  
sidering the figure 2, it is only necessary to express the principal  
directions of the wave movement in terms of space coordinates.

They're not just not

$$\frac{-(T)E + (T)D}{(9)(10) - 1} + \frac{-(\sqrt{T})E + (\sqrt{T})D}{(10) - 1}$$

[illegible]

energy may be associated with:

dissolution energy effect, and hence dissipation rate and kinetic  
number of developing voids theory is similar to that used for the  
growth, another theory of growth can be developed. Other two

occur as a result of local elastic energy rather than dissolution

$$= 27 = (27) + (27) + (27) - (27 + 27 + 27 + 27)$$

$$- (1 - \epsilon) \left( \frac{1}{2} \left( (1 - \epsilon)^2 + (1 - \epsilon) + (1 - \epsilon)^2 \right) - \frac{1}{2} \left( (1 - \epsilon)^2 + (1 - \epsilon) + (1 - \epsilon)^2 \right) \right)$$

$\phi = \frac{1}{2}(\theta + \psi)$

$$5 = \sqrt{(1,1)^2 + (1,1)^2 + (1,1)^2 + (1,1)^2} = \sqrt{4} = 2$$

This equation may be used in applications where the normal stresses rather than the principal stresses are known. For such cases, it is only necessary to substitute for the values of principal stresses in terms of components in Figure 3 to conform with the stated problem. Making this substitution, the equation becomes:

$$\sigma_e = \sqrt{(\tau + \tau_v)^2 + 2(1+\mu)(\tau + \tau_v)^2} - (1 - \sigma_e/\sigma_{yp}) \sqrt{\tau^2 + 2(1+\mu)\tau^2}$$

To summarize the above results, the equations for failure as depicted for the first two cases will be given in order to better compare them with test data,

Case (a) - Axial stress varying between maximum and minimum values while the shear stress remains constant. Failure will be predicted according to the theories as follows:

Shear Theory 
$$\sigma_e = \sqrt{(\sigma_e/\sigma_{yp}(\tau + \tau_v))^2 + 4(\sigma_e/\sigma_{yp})(\tau^2)}$$

Distortion Energy Theory 
$$\sigma_e = \sqrt{(\tau + \tau_v)^2 + 3\tau^2} - (1 - \sigma_e/\sigma_{yp})\sqrt{\tau^2 + 3\tau^2}$$

Strain Energy Theory 
$$\sigma_e = \sqrt{(\tau + \tau_v)^2 + 2(1+\mu)\tau^2} - (1 - \sigma_e/\sigma_{yp})\sqrt{\tau^2 + 2(1+\mu)\tau^2}$$

Case (b) - Axial and shear stress varying between maximum and minimum values of different magnitudes and in phase. Failure will be predicted according to the theories as follows:



This equation may be used in calculations where the normal stresses  
 between the principal stresses are known. For each stress, it  
 is only necessary to substitute for the value of principal stresses  
 in terms of components in Figure 2 to compare it to the stated  
 problem. Making this substitution, the equation becomes:

$$\sigma_c = \frac{(\sigma_1 + \sigma_2) \pm \sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}{2}$$

To determine the angle  $\theta$ , the equation for  $\tan 2\theta$   
 as developed for the first two cases will be given in order to  
 better compare them with each other.

Case (a) - Axiel stress varying between maximum and minimum values  
 while the shear stress remains constant. Failure will be pre-  
 dicted according to the theories as follows:

Strain Energy Theory  $\sigma_c = \frac{(\sigma_1 + \sigma_2) \pm \sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}{2}$

Maximum Shear Stress Theory  $\sigma_c = \frac{(\sigma_1 + \sigma_2) \pm \sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}{2}$

Strain Energy Theory  $\sigma_c = \frac{(\sigma_1 + \sigma_2) \pm \sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}{2}$

Case (b) - Axiel and shear stress varying between maximum and  
 minimum values at different locations and is given. Failure will  
 be predicted according to the theories as follows:



Shear Theory

$$\sigma_e = \sqrt{\left[ (\sigma_e / \sigma_{yp}) \sigma + \tau_v \right]^2 + 4 \left( \tau_e / \tau_{yp} + \tau_v \right)^2}$$

Distortion Energy Theory

$$\sigma_e = \sqrt{(\sigma + \sigma_v)^2 + 3(\tau + \tau_v)^2} - (1 - \sigma_e / \sigma_{yp}) \sqrt{\sigma^2 + 3\tau^2}$$

Strain Energy Theory

$$\sigma_e = \sqrt{(\sigma + \sigma_v)^2 + 2(1 + \mu)(\tau + \tau_v)^2} - (1 - \sigma_e / \sigma_{yp}) \sqrt{\sigma^2 + 2(1 + \mu)\tau^2}$$

Since three different theories have been presented and appear justified for design, the next important question to be answered is: which of the three theories should be used in the design of shafting? In order to answer this question, it is necessary to rely on experimental test data, of which there is little for the problem under discussion.

To date, it appears that for ductile materials, the distortion energy theory more closely approaches test data than the maximum shear theory or strain energy theory. This is indicated by tests in fluctuating bending and torsion with complete stress reversals made by Gough and interpreted by Marin (6), Figure 4. In the plot of the test data  $\sigma_1$  and  $\sigma_2$  are the principal stresses and  $\sigma_e$  the endurance limit for simple tension and compression. The stress ratios  $\sigma_1 / \sigma_e$  and  $\sigma_2 / \sigma_e$  at which failure occurs under combined bending and torsion are indicated by small circles and dots. Experiments by Lea and Budgen, Figure 5 and Figure 6, were made on circular specimens subjected to static torque and completely reversed bending stress. Here again, it is of interest to note that the distortion energy theory more closely approaches test data than do the other theories.

be the other theories.

the elastic energy theory more closely approaches test data than

tested bending theory. Here again, it is of interest to note that

elastic specimens subjected to static torque and completely re-

periments by Lee and Wedgwood, Figure 3 and Figure 6, were made on

beams and torsion are indicated by small circles and dots. Ex-

trinsic  $\epsilon_{1/2}$  and  $\epsilon_{2/2}$  at which failure occurs under combined

the endurance limit for static tension and compression. The stresses

of the first axis  $\epsilon_1$  and  $\epsilon_2$  are the principal stresses and  $\epsilon$

made by Gough and interpreted by Davis (8), Figure 4. In the plot

in fluctuating bending and torsion with constant stress reversals

where theory or strain energy theory. This is indicated by these

energy theory more closely approaches test data than the maximum

To date, it appears that for elastic materials, the elastic

limit for the elastic under discussion.

necessary to rely on experimental data, or which there is

design of materials? In order to answer this question, it is

assumed is: which of the three theories should be used in the

over the limit for design. The most important question to be

Since these different theories have been presented and ex-

$$\epsilon_c = \sqrt{(\epsilon_1 + \epsilon_2)^2 + 3(\epsilon_1 - \epsilon_2)^2} - (1 - \epsilon_c(\epsilon_1 + \epsilon_2)) \sqrt{\epsilon_1^2 + \epsilon_2^2}$$

Elastic Energy Theory

$$\epsilon_c = \sqrt{(\epsilon_1 + \epsilon_2)^2 + 4(\epsilon_1 - \epsilon_2)^2} - (1 - \epsilon_c(\epsilon_1 + \epsilon_2)) \sqrt{\epsilon_1^2 + \epsilon_2^2}$$

Strain Energy Theory



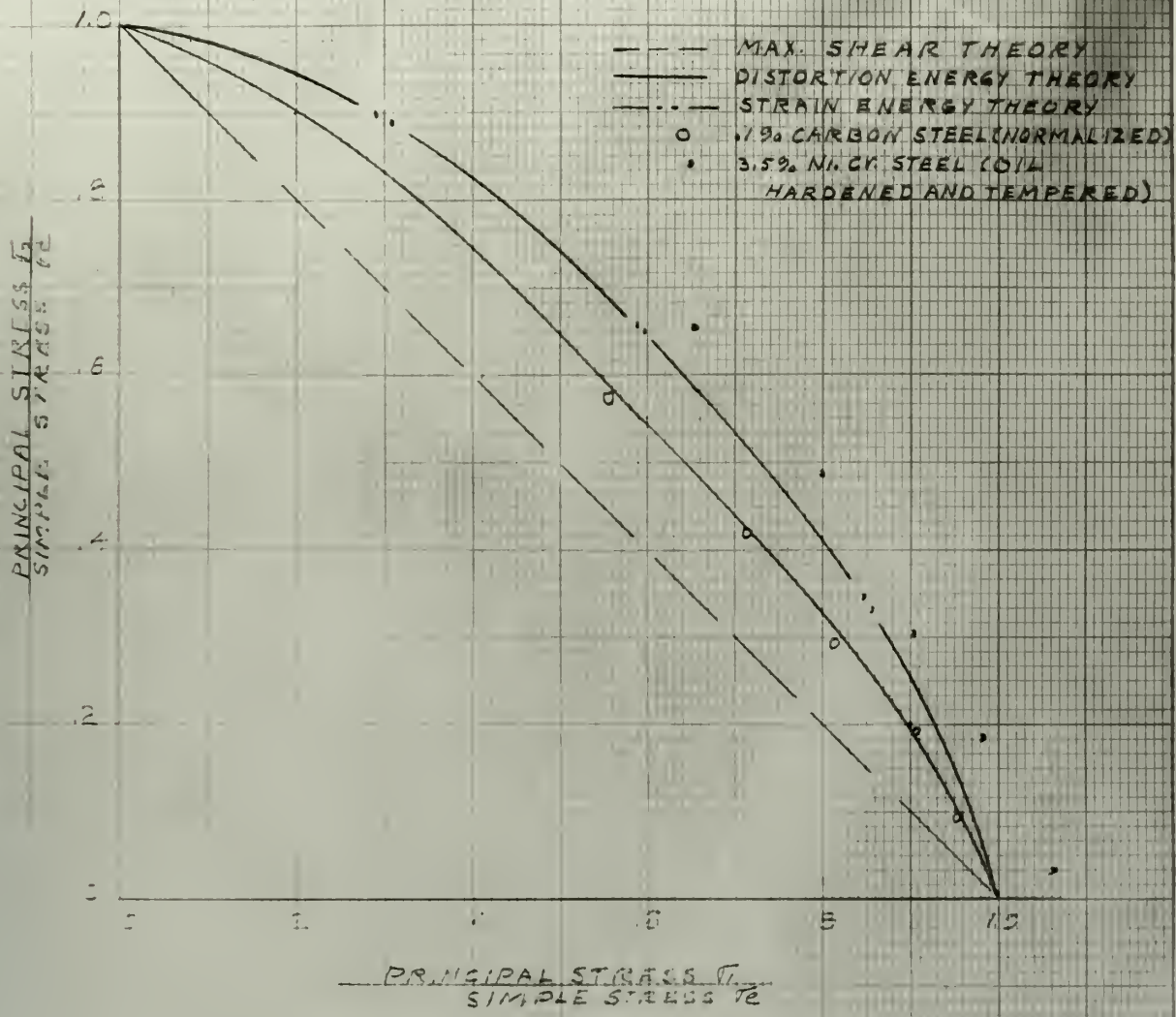
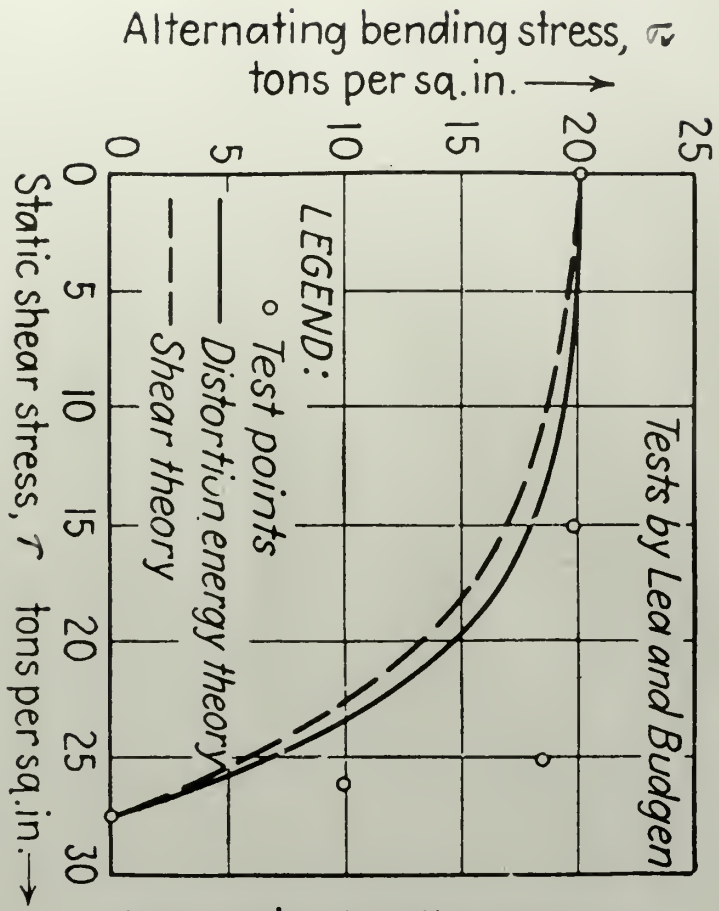


FIG. 4

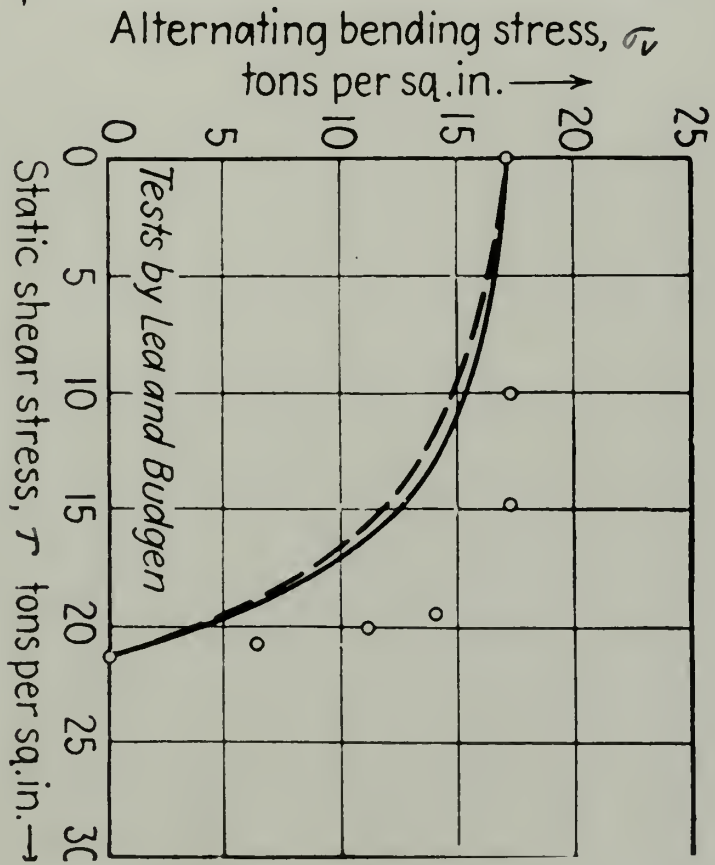






(a)

Tests on Ni-Cr steel  
(0.35% C, 0.6% Cr, 3.25% Ni)



(b)

Tests on "mild" steel  
(0.14% carbon)

Fig. 5-4 - Combined fatigue-stress results for ductile metals.



Of interest at the present time are experiments conducted by Gough (7) which more closely approximate conditions set forth in the problem under discussion. His work consisted of determining an empirical equation for variable bending and torsion of different magnitudes in phase. The empirical expression (in terms of the problem under discussion) Gough presented for failure is:

$$(\tau_v/\tau_e)^2 + (\tau_v/\tau_e)^2 (\tau_e/\tau_e - 1) + \tau_v/\tau_e (2 - \tau_e/\tau_e) = 1$$

Where  $\tau_e$  is the endurance limit in pure bending,  $\tau_e$  is the endurance limit in shear. As this equation is only developed for one material, it is of little assistance in present design, but indicates that research is underway to endeavor to determine a more exact expression for failure under condition of combined fatigue stress.

Since some basis for calculation of allowable stress under the assumed conditions has been established by the development of theories, to avoid failure in actual design, the stress imposed on shafting by operating loads must be less than this stress. To determine the relationship between the allowable stress, or working stress, in place of the stress components at failure, the axial stress at failure  $\tau_e$  is replaced by its working stress value  $\tau_e/h$  where  $h$  is the factor of safety.

The problem of choosing an adequate factor of safety in design is influenced by many factors. Some of these factors are:

1. Deficiencies in the theories of failure;
2. Assumption that materials are perfectly elastic, homogeneous, isotropic, and adhere to Hooke's Law;

Of interest at the present time are experiments conducted by Gough (7) which were closely approximate conditions and forth in the present under discussion. His work consisted of determining an empirical equation for variable bending and torsion of different materials in phase. The empirical expression in terms of the problem under discussion (Gough presented for failure is:

$$\left( \frac{T}{T_c} \right)^2 + \left( \frac{T}{T_c} \right)^2 \left( \frac{T}{T_c} - 1 \right) + \left( \frac{T}{T_c} \right)^2 (2 - \frac{T}{T_c}) = 1$$

Where  $T_c$  is the endurance limit in pure bending,  $T_c$  is the endurance limit in shear. As this equation is only developed for one material, it is of little assistance in present design, but indicates that research is underway to endeavor to determine a more exact expression for failure under condition of combined fatigue stress.

Since some basis for calculation of allowable stress under the assumed conditions has been established by the development of theories, to avoid failure in actual design, the stress imposed on shafting by operating loads must be less than this stress. To determine the relationship between the allowable stress, or working stress, and the stress components at failure, the axial stress at failure  $T_c$  is replaced by its working stress value  $T_c/n$  where  $n$  is the factor of safety.

The problem of choosing an adequate factor of safety in design is influenced by many factors. Some of these factors are:

1. Deficiencies in the theories of failure;
2. Assumption that materials are perfectly elastic,

homogeneous, isotropic, and known to Hooke's Law;



3. Machinery and fabrication errors;

4. Time effect, that is, no allowance is made in the determination of working stress for the deterioration of material with time due to corrosion, creep, electrolytic action, etc.;

5. Loads are rarely known accurately.

In consideration of these factors, it can be realized that the factor of safety must be greater than one (1), that is, the design must be overbalanced in favor of strength. Factors of safety are generally based on judgment and experience of the designer. For shafting, for example, the factor of safety has been judged to be between 2.25 and 3.00.

Since the method of determining the working stress has been considered, the equations defining failure by the various theories can be obtained. For comparison with static states of stress, both sides of the equation will be multiplied by  $\sigma_{yp}/\sigma_c$  such that working stresses defined by the various theories are:

Shear Theory

$$\sigma_w = \frac{\sigma_{yp}}{n} = \sigma_{yp}/\sigma_c \left[ \sqrt{(\sigma_c/\sigma_{yp} \sigma + \tau_v)^2 + 4(\tau \sigma_c/\sigma_{yp} + \tau_v)^2} \right]$$

Distortion Energy Theory

$$\sigma_w = \frac{\sigma_{yp}}{n} = \sigma_{yp}/\sigma_c \left[ \sqrt{(\sigma + \sigma_v)^2 + 3(\tau + \tau_v)^2} - (1 - \sigma_c/\sigma_{yp}) \sqrt{\sigma^2 + 3\tau^2} \right]$$

Strain Energy Theory

$$\sigma_w = \frac{\sigma_{yp}}{n} = \sigma_{yp}/\sigma_c \left[ \sqrt{(\sigma + \sigma_v)^2 + 2(1 + \mu)(\tau + \tau_v)^2} - (1 - \sigma_c/\sigma_{yp}) \sqrt{\sigma^2 + 2(1 + \mu)\tau^2} \right]$$

# 2. Summary and Conclusion

4. The effect of the deformation is such that

the deformation of the material is such that

material with the same deformation, strength, plasticity, etc.

1955

5. The effect of the deformation is such that

In comparison of these factors, it can be seen that

the factor of safety is such that it is greater than one (1), that is, the

factor of safety is such that it is greater than one (1), that is, the

safety is generally based on judgment and experience of the

designer. For example, the factor of safety has

been found to be between 2.25 and 3.00.

Since the method of determining the working stress has been

considered, the equation defining failure by the various theories

can be obtained. For comparison with elastic states of stress,

both sides of the equation will be multiplied by  $\sqrt{1/2}$  so that

working stresses defined by the various theories are:

## Stress Theory

$$\sigma_w = \frac{\sigma_y}{n} = \frac{\sigma_y}{n} \left[ \frac{(\sigma_x + \sigma_y)^2 + (\sigma_x - \sigma_y)^2}{2(\sigma_x + \sigma_y)^2} \right]$$

## Distortion Energy Theory

$$\sigma_w = \frac{\sigma_y}{n} = \frac{\sigma_y}{n} \left[ \frac{(\sigma_x + \sigma_y)^2 + (\sigma_x - \sigma_y)^2}{2(\sigma_x + \sigma_y)^2} \right]$$

## Strain Energy Theory

$$\sigma_w = \frac{\sigma_y}{n} = \frac{\sigma_y}{n} \left[ \frac{(\sigma_x + \sigma_y)^2 + (\sigma_x - \sigma_y)^2}{2(\sigma_x + \sigma_y)^2} \right]$$

## CHAPTER V

In the design procedure presented in the previous chapter, the assumption was made that the shaft was of constant cross section or gradually varying cross section and that the shaft did not contain any discontinuities. However, no mention was made of any particular characteristics of the material such as hardness or the effect of surface treatment. In presenting the effects of these items on shafting, the topics will be taken up separately dividing the chapter into three parts.

### PART I

#### STRESS CONCENTRATION

The effect of stress concentrations on metals subjected to alternating stress is of importance to engineers designing shafting because stress concentrations are invariably present due to fillets, holes, keyways, etc.

In the mathematical analysis of stress concentration based on the theory of elasticity for static loads, the results are usually stated in terms of a theoretical stress concentration factor:

$$K = \frac{\text{maximum stress of the section}}{\text{nominal stress of the section}}$$

since it is a function only of the geometry of the member for a



## SECTION V

In the design process presented in the preceding chapter, the assumption was made that the shaft was of constant stress section or gradually varying stress section and that the shaft did not contain any discontinuities. However, in practice, the shafts are made of many materials characterized by their material properties and the effect of surface treatments. In designing the shafts of these types or shafts, the design will be taken up separately dividing the shaft into these parts.

## PART I

### STRESS CONCENTRATION

The effect of stress concentration on shaft subjected to alternating stress is of importance to engineers designing shafts. The purpose stress concentration is usually present due to fillets, holes, keyways, etc.

In the mathematical analysis of stress concentration based on the theory of elasticity for static loads, the results are usually stated in terms of a theoretical stress concentration factor:

$$K = \frac{\text{Maximum stress of the section}}{\text{Nominal stress of the section}}$$

since it is a function only of the geometry of the member for a



specific loading condition.

If the loads acting are alternating, the stress concentration factor can no longer be defined as above because test results show that the full effect of the theoretical stress concentration factor is realized in only a limited number of cases. The decrease of strength brought about by discontinuities is stated in terms of a fatigue stress concentration factor.

$$k = \frac{\sigma_e}{\sigma_e'} = \frac{\text{ordinary endurance limit without stress concentration}}{\text{endurance limit with stress concentration effect}}$$

Peterson (8) in a discussion of stress concentration, suggests another way of presenting this factor which is the percentage decrease (d) of endurance strength due to stress concentration:

$$d = \frac{\sigma_e - \sigma_e'}{\sigma_e} \times 100$$

To find some basis for "K" being less than "k", Peterson (8) evaluated the principal of "stress concentration index" or sensitivity index which he expressed as the ratio:

$$q = \frac{k-1}{K-1}$$

The value of "q" ranges from zero (when  $k = 1.0$  for certain tests of cast iron) to unity (should the fatigue stress concentration factor "k" attain the theoretical value "K").

There are many factors that influence the magnitude of the stress concentration effect in the case of fatigue. It has been

specific loading conditions.

If the above action is assumed, the stress concentration factor can no longer be defined as above because test results show that the full effect of the theoretical stress concentration factor is realized in only a limited number of cases. The degree of strength brought about by discontinuities is stated in terms of a fatigue stress concentration factor.

$$K = \frac{\text{ordinary unnotched limit stress}}{\text{endurance limit at stress concentration effect}}$$

Peterson (3) in a discussion of stress concentration, suggests another way of presenting this factor which is the percentage decrease (d) of endurance strength due to stress concentration:

$$d = \frac{\sqrt{c} - \sqrt{r}}{\sqrt{c}} \times 100$$

To find some basis for "K" being less than "d", Peterson (3) revised the principle of "stress concentration index" or "stress intensity factor" which he expressed as the ratio:

$$p = \frac{K-1}{K}$$

The value of "p" ranges from zero (when  $K = 1.0$  for certain tests of cast iron) to unity (should the fatigue stress concentration factor "K" attain the theoretical value "K").

There are many factors that influence the magnitude of the stress concentration effect in the case of fatigue. It has been



found by tests that while theoretical stress concentration factors are independent of the material, as it conforms to Hooke's Law, test data shows that fatigue stress concentration factors are not. Likewise, it appears that the variation of "k" for similar test pieces of different materials cannot be correlated with any of the ordinary properties of materials such as ductility and hardness. Although for steels, Peterson (8) presented test data which indicates a possible relationship existing between fatigue stress concentration factors and ultimate strength. Another important effect is the size of the discontinuity. If the material and size of a specimen or shaft are kept constant and the size of the discontinuity is varied, the theoretical stress concentration factor decreases as the size decreases. Fatigue stress concentration factors show a similar tendency except for a marked decrease for very small discontinuities. Still another important factor in the determination is the effect of size. Peterson found for various types of steel that there is very little variation in endurance limit in geometrically similar specimens without discontinuities. However, for geometrically similar specimens having holes, fillets, or artificial cracks, it was determined that small specimens have higher endurance limits than larger ones. This indicates a lower stress concentration factor for small elements. For example, in the case of circular shafts, with a transverse hole, of .45% carbon steel with a ratio of hole diameter to shaft diameter of 0.0625, the stress concentration factor determined experimentally for reversed bending increased from 1.33 to 1.84 when the shaft diameter was increased from .5 to 3.0 inches.

With the knowledge that fatigue stress concentration factor

found by tests that while electrical tests concentrated factors  
are independent of the material, as is pointed out by the  
test data which shows that concentration factors are not  
likewise, it appears that the variation of the material itself  
effects of different materials should be considered with any of the  
ordinary formulas of analysis with an accuracy and precision  
although for elastic behavior (B) the material test data within in-  
dicates a possible relationship between the material stress  
concentration factors and ultimate strength. Another possible  
effect is the size of the discontinuity. If the material and size  
of a specimen or shaft are kept constant and the size of the dis-  
continuity is varied, the theoretical stress concentration factor  
decreases as the size decreases. Indirect stress concentration  
factors show a similar tendency except for a marked decrease for  
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However, for geometrically similar specimens having holes, fillets,  
or artificial cracks, it was determined that small specimens have  
higher endurance limits than larger ones. This indicates a lower  
stress concentration factor for small specimens. For example, in  
the case of circular shafts, with a transverse hole of 1/16  
inches steel with a nominal hole diameter of 0.015 inches of  
0.025 inches. The stress concentration factor determined experimentally  
for reversed bending increased from 1.35 to 1.84 when the hole  
diameter was increased from 0.015 to 0.025 inches. With the knowledge that fatigue stress concentration factor



can be determined from test data, the important question is how should these factors be applied in design? Since Nadai (1) determined that materials have sufficient elasticity to allow for localized yielding under static loads, it may be assumed that stress concentration effects only the fatigue stress. Tests by Peterson verify this assumption. Therefore, in determining the working stress, stress concentration should be applied to the alternating component of stress and not to the static component. Since in this paper working stress is determined from the values of maximum and mean combined stresses, the effect of stress concentration is to leave the mean stress unchanged and to increase the alternating component, thereby increasing the maximum stress.

For actual values to use in design, references such as Lipson, Noll, and Clock (9), or Roark (10) may be consulted.

## PART II

### HARDNESS

Since endurance limits, which are a measure of the allowable stress under fatigue conditions, for all types of materials are unknown, a search has been made by engineers to determine if some correlation can be made between this property of metals and other properties that can be measured with comparative ease.

Among the mechanical properties that can be used to give a good estimate, hardness is considered by many to be the most

For actual values to be in better reference with the  
the alternative dependent, thereby increasing the current stress,  
capitalism is to leave the main stress unchanged and to increase  
of nature and man combined stresses, the effect of which con-  
Since in this paper working stress is distinguished from the values  
technical development of stress and not to the stress component.

## II DAS

## 205 MENA

Good estimate, however, is considered by many to be the most  
among the mechanical properties that can be used to give a  
properties that can be measured with comparative ease.  
correlation can be made between this property of metals and other  
unknown, a search has been made by engineers to determine if some  
stress under fatigue conditions, for all types of materials are  
Since strength limits, which are a measure of the allowable



valuable for steels. Considering non-ferrous metals, there is too much scatter of results to justify any correlation for estimating endurance limit.

The one factor that makes this possible is that for static loading tests, the relationship between hardness and tensile strength, reference (11), is represented by a band that includes test data on alloys as well as plain carbon steels. The band width for all practical purposes can be considered negligible. That is, hardness may be considered proportional to the tensile strength.

In the case of endurance tests, the relationship between endurance limit and hardness was further complicated by the fact that tests showed that endurance limit did not only depend on hardness, but is also a function of the quality of the surface. For example, Hankins, Becker, and Mills (12), indicated that the variation in endurance limit between specimens finished with fine and coarse emery paper was 3% for a steel of 118,000  $\psi$ s, tensile strength and 11% for a steel of 138,000  $\psi$ s, tensile strength; and Hager (13) found no difference in endurance limit between rough and finished machined specimens of soft steel, the endurance limit of both being 10% less than for polished specimens. From a group of such tests (considering the fact that tensile strength is approximately proportional to hardness), Lipson, Noll, and Clock (9) devised a set of curves, Figure 7, showing the variation of endurance limit with hardness for materials whose surface conditions are in four groups; ground, machined, hot rolled, and forged. The ground surface finish includes ground, honed, lapped, and super-finished, and the machined surface finish includes rough and

The one factor which seems to be peculiar to this type of  
 loading test, the relationship between hardness and tensile  
 strength, reference (11), is represented by a curve showing  
 test data of alloys as well as plain carbon steels. The same relationship  
 for all practical purposes can be considered applicable. That is,  
 hardness may be considered proportional to the tensile strength.  
 In the case of numerous steels, the relationship between  
 hardness and tensile strength was further established by the fact  
 that tests showed that hardness itself did not only depend on  
 hardness, but it also a function of the quality of the material.  
 For example, Machine, 1045, and 1045 (12), indicate that the  
 variation in hardness itself between specimens finished with fine  
 and coarse sand, about the 12 for a steel of 128,000 psi tensile  
 strength and 12 for a steel of 128,000 psi tensile strength; and  
 Figure (13) found no difference in hardness limits between rough  
 and finished machine specimens of both steel, the hardness limit  
 of both being 1045 less than the polished specimens. From a group  
 of such tests considering the fact that tensile strength is ap-  
 proximately proportional to hardness, (Figure 10, 11, and 12) it  
 devised a set of curves. Figure 1, showing the variation of ap-  
 proximate limit with hardness for various types of surface conditions  
 are in four groups; ground, machined, hot rolled, and forged. The  
 ground surface which includes ground, wheel, tapered, and super-  
 finished, and the machined surface which includes rough and



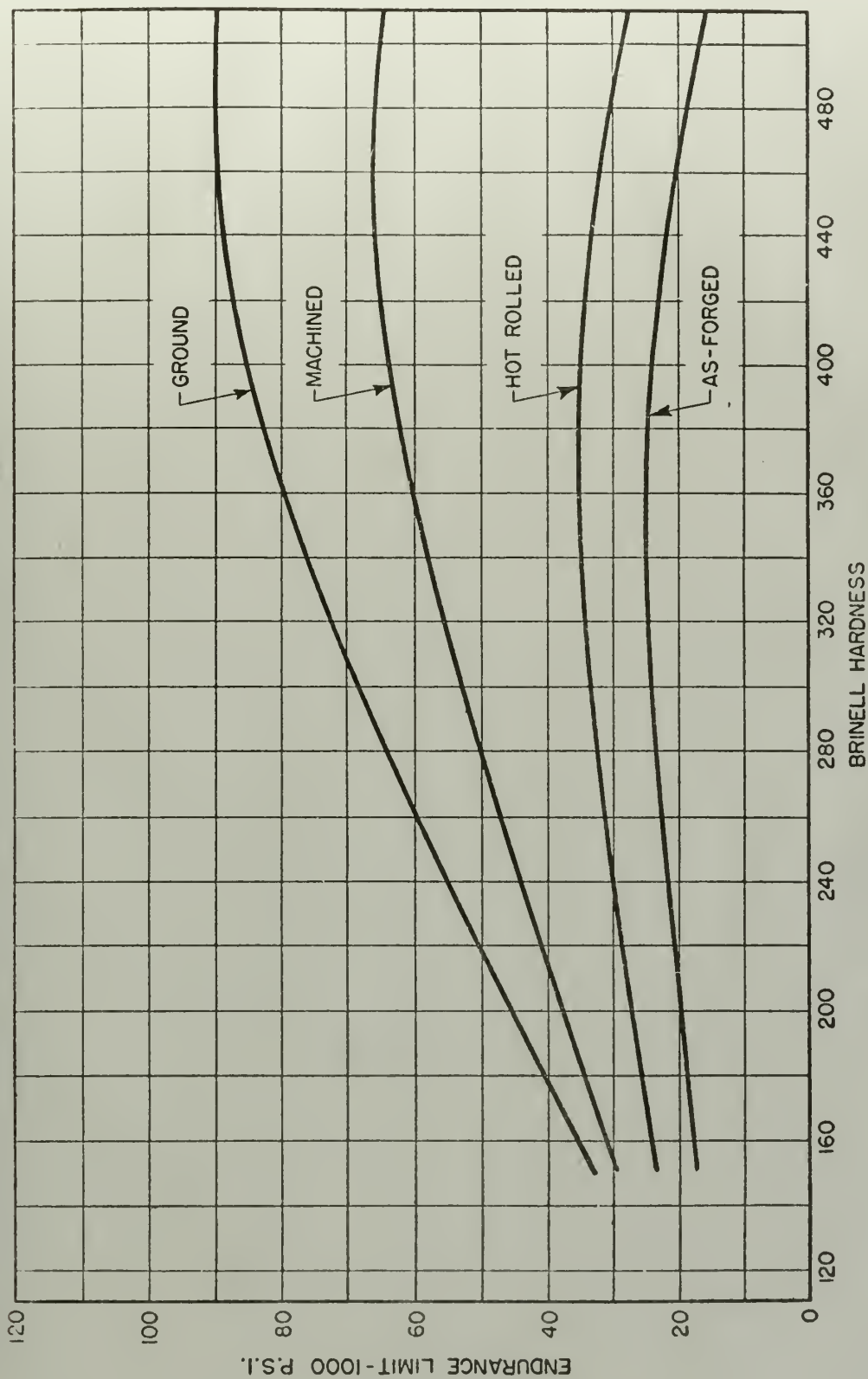


FIG. 76. Relationship between hardness and strength for fatigue loading.



finished machined. Further evidence of the validity of these curves is given in tabular form by Eddy (9) for Brinell hardness ranging from 160 to 555.

### PART III

#### SURFACE TREATMENT

Another factor which influences attainable stress is surface treatment. In most cases, the treating process is applied to metals to improve such properties as wear, corrosion resistance, etc.

The commonly used processes whose effects have been studied are cold working and surface hardening. These two processes will be considered separately.

#### COLD WORKING:

In cold working, the material is strained beyond the yield point and caused to flow plastically. The outstanding effects on metals are to raise the elastic limit markedly, noticeably increase the tensile stress, and decrease the ductility.

Cold working of metals appears to exert two opposing effects: (1) It elongates the crystalline grains in the direction of working into a more favorable position for resisting slip and fracture; (2) It tends to start new minute fractures, or tends to

finished material. Further evidence of the validity of these curves is given in another paper by May (1) for small particles ranging from 100 to 500.

### PAGE III SUBJECT TREATMENT

Another factor which influences substantially stress is surface treatment. In most cases, the treating process is applied to metals to improve such properties as strength, corrosion resistance, etc. The commonly used processes whose effects have been studied are cold working and surface hardening. These two processes will be considered separately.

#### COLD WORKING:

In cold working, the material is strained beyond the yield point and strained to the plastic limit. The outstanding effects on metals are to raise the yield strength, increase the hardness, increase the tensile strength, and decrease the ductility. Cold working of metals appears to exert two opposing effects: (1) It elongates the crystalline grains in the direction of working into a more favorable position for resisting slip and fracture; (2) It tends to start new stress fractures, or leads to



set up severe internal stresses in the metal making fracture possible by a small additional applied stress. That is, for some degree of cold working, there is a maximum net benefit while for a more severe degree of cold working, the damage done increases more rapidly than does the benefit.

The most commonly used methods of cold working are cold rolling, stretching, and shot peening. The effect of all three of these processes appears to be an increase in the fatigue limit of materials. The amount of increase is usually a function of the condition of the surface before the process is applied. A variety of results has been obtained for the effect of cold stretching on the fatigue limit for non-ferrous metals. Tests (3) of brass and copper rod in which there is a reduction of area of 55% in a single pass of the cold drawing process showed no increase in fatigue limit over the limit of the same metal hot rolled. Tests of nickel and of other non-ferrous metals subjected to less drastic reduction, than that mentioned above, showed an appreciable increase in fatigue limit over the same metal annealed. For detailed data on non-ferrous metals, references such as Moore and Kommers (3) or Metals Handbook may be consulted.

For ferrous metals, the same general trend is noted. From a compilation of data (9), a general observation appears to be that, for any type of cold working, the minimum increase of fatigue limit is 2% for a polished hardened alloy after shot peening to 150% for a cold rolled machined specimen (SAE 1045 steel) after normalizing. Although insufficient data are available to make specific conclusions

and in some cases, the amount of cold working is not known. This is, for some, possible by a small amount of cold working. This is, for some, better of cold working, there is a maximum and minimum for a more severe degree of cold working, the degree of cold working is more readily than for the metal.

The most commonly used method of cold working is cold rolling. The effect of cold rolling is to increase the strength of the material. The amount of increase is usually a function of the condition of the surface before the process is applied. A variety of results have been obtained for the effect of cold rolling on the fatigue limit for non-ferrous metals. Table (3) of stress and strain for which there is a reduction of size of 50% in a single pass of the cold drawing process shows an increase in fatigue limit over the limit of the same metal but rolled. Table of nickel and of other non-ferrous metals subjected to less drastic reduction, than that mentioned above, shows an appreciable increase in fatigue limit over the same metal annealed. For detailed data on non-ferrous metals, references such as Moore and Roberts (2) or Metals Handbook may be consulted.

For ferrous metals, the same general trend is noted. From a compilation of data (3), a general observation appears to be that for any type of cold working, the fatigue limit of rolled steel is 25% for a rolled material after cold working to 100% for a cold rolled material (see Table 1044 steel) after normalizing. Although insufficient data are available to make specific comparisons



on the beneficial effect of cold working on fatigue limit, the general trend can be noted in the case of shot peening. In the case of un-notched specimens, the increase in the fatigue limit for polished specimens seldom exceeds 20%, and in many cases, is less than 10%. For hot rolled specimens, the corresponding increase is 30 to 50% and for as-forged parts, 100%. The general trend is also apparent in the case of notched specimens, although the percentage improvement is higher.

#### SURFACE HARDENING:

Surface hardening is a process which increases the hardness of the surface to a depth ranging from a few thousandths of an inch to  $\frac{1}{4}$  of an inch or more. Only a comparatively few number of tests have been conducted. They all show an increase in the endurance limit for ferrous and non-ferrous metals.

Remembering that endurance limit may be determined from hardness (Part II), Lipson, Noll, and Clock (9), presented a method for estimating endurance limits for surface treated ferrous parts. The method is based on the premises that, through hardness distribution over any cross section is non-uniform, the hardness distribution may be thought of as consisting of a hardened case and soft core. With this in mind to determine whether the case or core hardness should be used, the method proposed consists of superimposing the applied stress on the allowable stress as estimated by hardness. By using this method, it was determined that for estimating endurance limits, the hardness of the case should be used for un-notched machine parts while the hardness of the core

of the experimental effect of cold working on fatigue life, but  
general trends can be noted in the case of both specimens. In the  
case of unnotched specimens, the increase in the fatigue life  
for polished specimens versus B01, was in every case, is  
less than 10%. For hot rolled specimens, the corresponding increase  
is 50 to 60% and for as-rolled (B01, B02). The fatigue life is  
also apparent in the case of notched specimens, although the per-  
centage improvement is slight.

### SURFACE HARDNESS

Surface hardness is a process which increases the hardness  
of the surface to a depth ranging from a few microns to an  
inch or more. Only a comparatively few number of  
tests have been conducted. They all show an increase in the sur-  
face life for ferrite and non-ferrite metals.  
Remembering that endurance limit may be determined from  
hardness (Part II, Lipson, Wolf, and Clock (6)), presented a  
method for determining endurance limits for surface treated ferrite  
steels. The method is based on the hardness test, through hardness  
distribution over any cross section is non-uniform, the hardness  
distribution may be thought of as consisting of a hardened zone  
and soft core. With this in mind to determine whether the case or  
core hardness should be used, the method proposed consists of super-  
imposing the expected effect of the hardness test as estimated  
by hardness. By using this method, it was determined that for  
estimating endurance limits, the hardness of the case should be  
used for on-colded machine parts while the hardness of the core



should be used for severely notched machine parts. Test data substantiated this method for carbonized un-notched specimens (9). It can be surmized that it will apply to induction hardened and flame hardened machined parts. However, because of the nature of processing, there is between the case and core a transformation zone that is essentially in a normalized or annealed state. Its hardness may be less than the hardness of the core. Therefore, for conservative design, the hardness of the transition zone, rather than the hardness of core, should be used for un-notched or mildly notched machine parts. In the presence of sharp notches, the hardness of the case should be used for carbonized machine parts.

In the case of nitriding, tests have indicated that the endurance limit is unaffected by surface finish for un-notched or moderately notched specimens. Only very sharp discontinuities show any decrease in endurance limit. As a result, the above considerations are not applicable for nitriding; thus, endurance limits experimentally determined should be used in design.

For other discussions on the effect of surface treatment, see Peterson and Lessells (14), Woodvine (15), and Hoger (16).

[illegible]

CHAPTER VI  
ILLUSTRATIVE PROBLEMS

In order to illustrate the use of the suggested concepts for the design of rotating circular shafts for cases (a) and (b), a group of problems is presented in which the diameter ( $d$ ) is the desired quantity. The loading will be restricted to pure bending and torque. The distortion energy theory of failure will be used throughout since it was suggested in Chapter IV that it more accurately predicts failure when shafting is subjected to combined fatigue stress. The material is assumed to be a S.A.E. 1035 steel which has been quenched in water at 1525-1575°F with a Brinell hardness of 212. Since accurate data for yield point stress  $\sigma_{yp}$  and endurance limit  $\sigma_e$  is not known, values are taken from reference (9) ( $\sigma_{yp} = 69000 \text{ psi}$ ,  $\sigma_e = 37000 \text{ psi}$ ). To show the effect of stress concentration, a profile keyway is considered. The values used for fatigue stress concentration factors are those given in reference (9). In each problem, the safety factor ( $n$ ) is considered to be 2.5. The rotating shaft is shown diagrammatically in Figure 7.

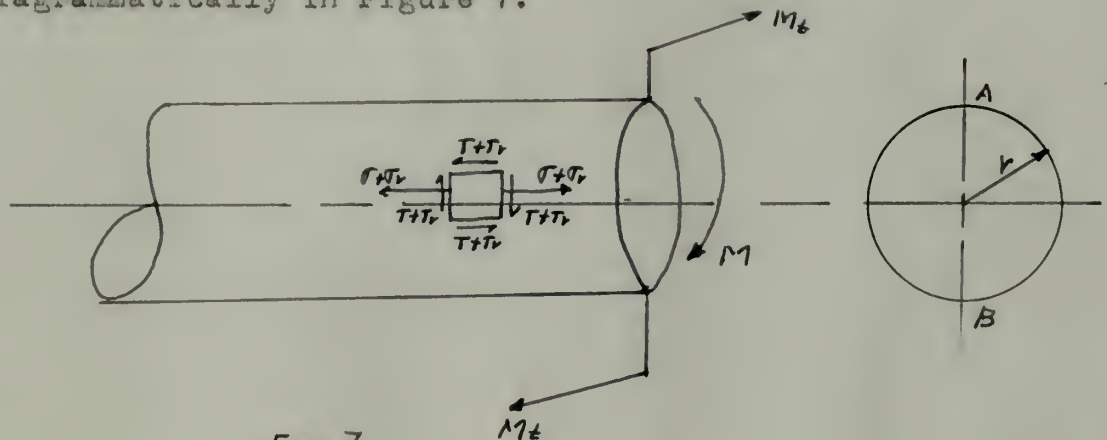


FIG. 7



# CHAPTER VI FATIGUE STRESS ANALYSIS

In order to illustrate the use of the proposed concepts for the design of rotating shafts under (a) and (b), a group of problems is presented in which the diameter (d) is the desired quantity. The loading will be restricted to pure bending and torsion. The distortion energy theory of failure will be used throughout since it was suggested in Chapter IV that it more accurately predicts failure when shearing is subjected to combined fatigue stress. The material is assumed to be a S.A.E. 1045 steel which has been quenched in water at 1500-1550°F with a Brinell hardness of B.H. 300. Accurate data for yield point stress  $\sigma_p$  and endurance limit  $\sigma_e$  is not known, values are taken from reference (1) ( $\sigma_p = 66000 \text{ psi}$ ,  $\sigma_e = 33000 \text{ psi}$ ). To show the effect of stress concentration, a profile keyway is considered. The values used for fatigue stress concentration factors are those given in reference (2). In each problem, the safety factor (n) is considered to be 2.0. The rotating shaft is shown schematically in Figure 7.

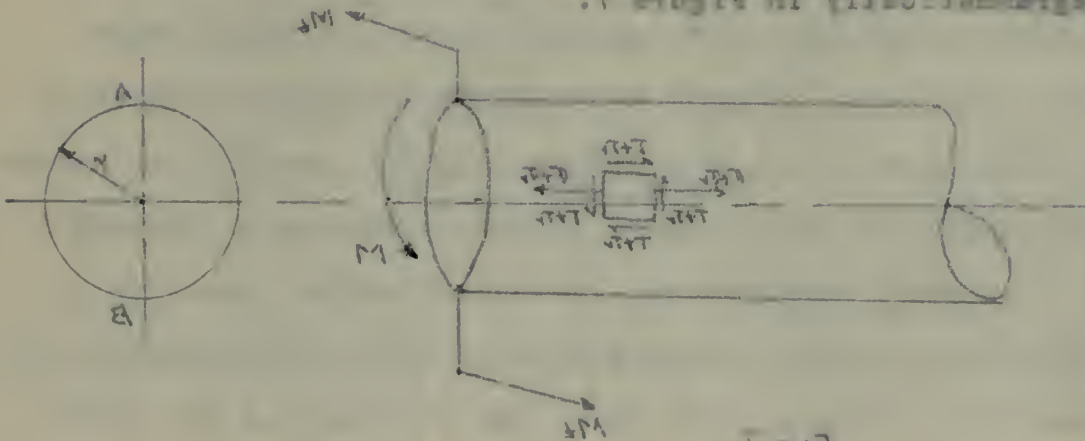


Fig. 7  
28



ROTATING CIRCULAR SHAFT SUBJECTED TO FLUCTUATING PURE BENDING  
AND STATIC TORQUE

Let the value of bending moment vary from a maximum,  $M' = 12000 \text{ lb in}$ , to a minimum,  $M'' = 8000 \text{ lb in}$ , while the torque remains constant,  $M_t = 6000 \text{ lb in}$

The normal stress produced by the moment  $M$  will vary as the shaft rotates, and as  $M$  changes from a maximum,  $M'r/I$  (point A, Figure 7), to a minimum,  $-M'r/I$  (Point B, Figure 7). The values of maximum and mean normal stresses are:

$$\sigma + \sigma_v = \frac{32 M'}{\pi d^3} = \frac{12.25 \times 10^4 \text{ psi}}{d^3}, \quad \sigma_v = 0$$

The shear stress produced by the torque remains constant; thus, the maximum and mean values of stress are:

$$\tau + \tau_v = \tau = \frac{16 M_t}{\pi d^3} = \frac{3.06 \times 10^4}{d^3}$$

Since the criteria of failure is:

$$\sigma_w = \frac{\sigma_e}{n} = \sqrt{(\sigma + \sigma_v)^2 + 3\tau^2} - (1 - \sigma_e/\sigma_{yp}) \sqrt{3\tau^2}$$

substituting the values given above into this expression, the diameter is determined to be

$$d = 1.95 \text{ in}$$

AND STRESS DISTRIBUTION

Let the value of bending moment per unit length,  $M$ , be constant,  $M = M_0$ , and the value of the twisting moment per unit length,  $T$ , be constant,  $T = T_0$ . The normal stress produced by the moment  $M$  will vary as the angle  $\theta$ , and as  $M$  changes from a maximum,  $M = M_0$ , (point A, Figure 7), to a minimum,  $M = 0$ , (point B, Figure 7).

The values of maximum and minimum stresses are:

$$\sigma_{\max} = \frac{M_0 y}{I} = \frac{M_0}{I} y, \quad \sigma_{\min} = 0$$

The shear stress produced by the torque  $T$  is constant:

Thus, the maximum and minimum values of stress are:

$$\sigma_{\max} = \frac{M_0 y}{I} = \frac{M_0}{I} y, \quad \tau_{\max} = \frac{T_0}{J}$$

Since the criteria of failure are:

$$\sigma_{\max} = \frac{M_0 y}{I} = \frac{M_0}{I} y, \quad \tau_{\max} = \frac{T_0}{J}$$

substituting the values given above into this expression, the

diameter is determined to be

$$d = 1.92 \text{ in}$$

ROTATING CIRCULAR SHAFT SUBJECTED TO FLUCTUATING PURE BENDING  
AND STATIC TORQUE WITH A PROFILE KEYWAY

Let the bending moment and torque remain the same as the previous problem. As a result, the only change in stress is that the maximum normal stress (same as the variable component as  $\sigma = \sigma$ ) becomes  $\sigma + K_b \sigma_v$ . This follows the concept developed in Chapter V, that is, the fatigue stress concentration factor is applied only to the variable stress. The value of  $K_b$  (fatigue stress concentration for pure bending) from reference (9) is 2.0.

Since the criteria of failure is:

$$\sigma_w = \frac{\sigma_e}{n} = \sqrt{(\sigma + K_b \sigma_v)^2 + 3\tau^2} - (1 - \sigma_e / \sigma_{yp}) \sqrt{3\tau^2}$$

substituting the values given into this expression, the diameter determined is  $d = 2.5$  in.

Note due stress concentration the required diameter has increased by 30% as compared to the previous problem.

STATISTICAL ANALYSIS OF MECHANICAL TEST RESULTS  
AND THE EFFECT OF TEMPERATURE ON THE STRESS-STRAIN CURVE

Let the yield point stress and ultimate stress be denoted by  $\sigma_y$  and  $\sigma_u$  respectively. As a result, the yield strength is defined as that the maximum normal stress (tension or compression) at which the material has not yet failed. This failure is usually defined in Chapter 9, that is, the failure stress concentration factor is applied only to the average stress. The value of  $K_t$  (failure stress concentration for pure bending) from reference (8) is 2.0.

Since the criteria of failure is:

$$\sigma = \frac{F}{A} = \frac{F}{A_0} \sqrt{\frac{A_0}{A}} = \frac{F}{A_0} \sqrt{1 + K_t} \quad (1)$$

substituting the values given into this equation, the diameter determined is 0.5 in. Note the stress concentration the negative diameter has increased by 50% as compared to the previous problem.



ROTATING CIRCULAR SHAFT SUBJECTING TO FLUCTUATING PURE BENDING  
AND FLUCTUATING TORQUE

Let the value of bending moment vary from a maximum  $M' = 12000 \text{ lb in}$ , to a minimum,  $M'' = 8000 \text{ lb in}$ , while the torque varies from a maximum,  $M_t = 6000 \text{ lb in}$ , to a minimum,  $M_t'' = 4000 \text{ lb in}$

The normal stress produced by the varying moment will remain the same as for the first two problems, namely:

$$\sigma + \sigma_v = \frac{32 M'}{\pi d^3} = \frac{12.25 \times 10^4}{d^3} \text{ lb/in}^2, \quad \sigma = 0$$

The shearing stress imposed by the torque will vary as the torque changes. The maximum mean and variable values of shear stress are:

$$T + T_v = \frac{16 M_t'}{\pi d^3} = \frac{3.06 \times 10^4}{d^3}, \quad T = \frac{16}{\pi d^3} \left( \frac{M_t' + M_t''}{2} \right) = \frac{2.55 \times 10^4}{d^3}$$

$$T_v = \frac{16}{\pi d^3} \left( \frac{M_t' - M_t''}{2} \right) = .51 \times 10^4$$

Since the criteria of failure is:

$$\sigma_w = \frac{\sigma_e}{n} = \sqrt{(\sigma + \sigma_v)^2 + 3(T + T_v)^2} - (1 - \sigma_e/\sigma_{yp}) \sqrt{3T^2}$$

substituting the values given into this expression, the diameter is determined to be  $d = 2.00$ .

# ROTATING CIRCULAR SHAFTS SUBJECTED TO TORSION

## AND BENDING MOMENTS

Let the value of bending moment vary from a maximum

$M_1 = 12000 \text{ lb-in.}$  to a minimum,  $M_2 = 8000 \text{ lb-in.}$  while the torque

varies from a maximum,  $T_1 = 6000 \text{ lb-in.}$  to a minimum,  $T_2 = 4000 \text{ lb-in.}$

The normal stress produced by the bending moment will be

equal to zero at the first two positions, namely:

$$\sigma + \tau = \frac{M_1}{I} = \frac{12000}{\frac{\pi d^4}{64}} = 12.22 \times 10^6 \text{ psi}$$

The shearing stress imposed by the torque will vary as the

torque changes. The maximum mean and principal values of stress

are given by:

$$\sigma + \tau = \frac{M_1}{I} = \frac{12000}{\frac{\pi d^4}{64}} = 12.22 \times 10^6 \text{ psi}$$

$$\tau = \frac{T}{J} = \frac{4000}{\frac{\pi d^4}{32}} = 2.22 \times 10^6 \text{ psi}$$

Since the principal stress is:

$$\sigma_m = \frac{\sigma}{2} \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{12.22 \times 10^6}{2} \pm \sqrt{\left(\frac{12.22 \times 10^6}{2}\right)^2 + (2.22 \times 10^6)^2}$$

substituting the values gives the following expression. The diameter

is determined to be 2.00

ROTATING CIRCULAR SHAFT SUBJECTED TO FLUCTUATING PURE BENDING  
AND FLUCTUATING TORQUE WITH A PROFILE KEYWAY

The conditions of loading are considered to remain the same as in the previous problem. The only changes in stress are that the maximum normal stress (same as the variable stress as  $\sigma = 0$ ) becomes  $\sigma + K_b \sigma_v$  and the maximum shear stress becomes  $\tau + K_s \tau_v$ . The application of the fatigue stress concentration factor follows the concept developed in Chapter V, fatigue stress concentration factor is applied only to the variable stress. The values of  $K_b$  (fatigue stress concentration factor for pure bending) and  $K_s$  (fatigue stress concentration factor for applied torque) are, from reference (9), 2.0 and 1.6 respectively.

Since the criteria of failure is:

$$\sigma_w = \frac{\sigma_e}{n} = \sqrt{(\sigma + K_b \sigma_v)^2 + 3(\tau + K_s \tau_v)^2} - (1 - \sigma_e / \sigma_{yp}) \sqrt{3\tau^2}$$

substituting the values given into this expression, the diameter is determined to be  $d = 3.0$ .

Note due stress concentration the diameter has increased 50% as compared to the previous problem.



The conditions of loading are considered to remain the same as in the previous problem. The only change in stress is that the maximum normal stress (now as the variable stress  $\sigma = 0$ ) increases  $\sigma + K_1 \sigma$  and the maximum shear stress decreases  $\tau + K_2 \tau$ . The modification of the fatigue stress concentration factor follows the concept developed in Chapter V. Fatigue stress concentration factor is applied only to the surface stress. The value of  $K_1$  (fatigue stress concentration factor for pure bending) and  $K_2$  (fatigue stress concentration factor for pure shear) are, from reference (9), 0.0 and 1.0 respectively.

Since the value of fatigue is:

$$\sigma_w = \frac{\sigma}{n} = \frac{\sigma}{(1 + K_1)^2 + 2(K_1 + K_2)^2} - (1 - \tau) \sqrt{3\tau}$$

substituting the values given into this expression, the diameter is determined to be  $n = 3.0$ .

Note that stress concentration the diameter has increased 50% as compared to the previous problem.



## CHAPTER VII

### CONCLUSIONS

Since to date, no exact solution is known for determining the failure of machine components subjected to fatigue stress, it is the author's opinion that the distortion energy theory gives the best approximation for varying tension, compression, and shear loads in phase.

From an analysis of the problem, the best analytical approach appears to be that stress concentration effects only the variable component of stress. Furthermore, the endurance limit seems to be a function of the hardness of, and surface treatment applied to, materials. Thus, it can usually be taken into the analytical approach depending on the previous history of the material.

Since the overall problem of fluctuating loads is of such great interest to designers, the writer feels that more research should be conducted in order to determine a more exact solution for the problem of combined fatigue stress.

Since in fact, no exact solution is known for determining the value of certain components subjected to fatigue stress, it is the author's opinion that the following theory gives the best approximation for varying constant, non-repeating, and other loads in phase.

From an analysis of the problem, the case analytical approach appears to be best since consideration effects only the variable component of stress. Furthermore, the modulus itself seems to be a function of the hardness of, and surface treatment applied to, materials. Thus, it can usually be taken into the analytical approach depending on the previous history of the material.

Since the overall problem of determining loads is of great interest to designers, the writer feels that some research should be conducted in order to determine a more exact solution for the problem of combined fatigue stress.

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